Transonic Integral Equation Formulation for Lifting Profiles and Wings

Pradip Niyogi*
Jadavpur University, Calcutta, India

Introduction

THERE are at present three different integral equation formulations for studying the direct problem of steady inviscid isentropic transonic flow past thin lifting profiles, where the freestream Mach number $M_\infty\,(<1)$ and the body shape are prescribed. Two of these formulations are by Nørstrud,[1,2] and the third is by Nixon and Hancock.[3] The formulation in Ref. 2 has been extended to the three-dimensional case of lifting wings by Nørstrud.[4] The basic integral equation governed by the perturbation potential of a lifting wing was given earlier by Heaslet and Spreiter[5] and Klunker.[6] About two years ago, a controversy started from a criticism by Nixon[7] regarding the correctness of the formulation of Nørstrud[2] which, according to Nixon, leads to nonunique solution. Nørstrud,[8] on the other hand, pointed out some mistakes in the Nixon[7] critique, creating further confusion. Recently, Chakraborty and Niyogi[9] showed that the formulations of Nørstrud[1] and Nixon and Hancock[3] are equivalent. The purpose of the present Note is to point out a sign error in the second formulation of Nørstrud[2] and to indicate an alternative way of obtaining the appropriate formulation for this case.

Nørstrud's Second Formulation

Nørstrud[2] established the following integral equations for the reduced U velocity components on the upper and lower side of the profile, denoted, respectively, by $U(X,+0)$ and $U(X,-0)$:

$$U(X,+0)=U_p(X,+0)-\frac{\Delta U(X)-\Delta U_p(X)}{2}+\lim_{Y\to+0}\frac{\partial I(X,Y;U)}{\partial x} \tag{1}$$

and

$$U(X,-0)=U_p(X,-0)+\frac{\Delta U(X)-\Delta U_p(X)}{2}+\lim_{Y\to-0}\frac{\partial I(X,Y;U)}{\partial x} \tag{2}$$

The reduced quantities Φ, U, X, and Y are defined in terms of the true quantities, denoted by the lower case letters:

$$M_\infty<1:\;\; \Phi(X,Y)=K(\varphi-u_\infty x-v_\infty y)/[(1-M_\infty^2)u_\infty] \tag{3a}$$

$$U(X,Y)=\frac{K}{1-M_\infty^2}\cdot\frac{u-u_\infty}{u_\infty}\qquad V(X,Y)=\frac{K}{(1-M_\infty^2)}\cdot\frac{v-v_\infty}{u_\infty} \tag{3b}$$

$$X=x,\quad Y=y\sqrt{1-M_\infty^2} \tag{3c}$$

The parameter K is a function of the freestream Mach number, for which different approximate values such as $M_\infty^2(\gamma+1)$ or $(1-M_\infty^2)/[(1/M_\infty^*)^{-1}]$ may be used. M_∞^* denotes critical freestream Mach number and u_∞, v_∞ freestream velocity components. The symbol Δ is used to denote the difference of the value of a quantity on the upper and lower profiles sides, so that

$$\Delta U(X)=U(X,+0)-U(X,-0)$$

$$\Delta U_p(X)=U_p(X,+0)-U_p(X,-0) \tag{4}$$

Here, $U_p(X,Y)$ denotes the known U component of the Prandtl velocity, being, in reduced coordinates, the solution of Laplace's equation

$$\Phi_{XX}+\Phi_{YY}=0 \tag{5}$$

satisfying the same boundary condition as the corresponding nonlinear problem. Furthermore, $I(X,Y;U)$ represents the double integral

$$I(X,Y;U)=\frac{1}{2\pi}\int_S\int\Phi_\xi(\xi,\eta)\Phi_{\xi\xi}(\xi,\eta)\ln[(\xi-X)^2+(\eta-Y)^2]^{1/2}\mathrm{d}S \tag{6}$$

where S denotes the whole space from which the singular point $\xi=X$, $\eta=Y$ has been removed by a small circle and the slit $0\le\xi\le1$, $\eta=\pm0$ has been excluded.

It should be noted that, if the Oswatitsch principal value definition[1] is used, according to which the singularity $\xi=X$, $\eta=Y$ is excluded by means of an infinitesimal slit parallel to the η axis, calculation shows that

$$\lim_{Y\to+0}\frac{\partial I(X,Y;U)}{\partial X}=\frac{U^2(X,+0)}{2}-\lim_{Y\to+0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{U^2(\xi,\eta)}{2}\cdot\frac{(\xi-X)^2-(\eta-Y)^2}{[(\xi-X)^2+(\eta-Y)^2]^2}\mathrm{d}\xi\mathrm{d}\eta \tag{7a}$$

and

$$\lim_{Y\to-0}\frac{\partial I(X,Y;U)}{\partial X}=\frac{U^2(X,-0)}{2}-\lim_{Y\to+0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{U^2(\xi,-\eta)}{2}\cdot\frac{(\xi-X)^2-(\eta-Y)^2}{[(\xi-X)^2+(\eta-Y)^2]^2}\mathrm{d}\xi\mathrm{d}\eta \tag{7b}$$

In pursuing the calculations of Nørstrud,[2] it came to our notice that the signs of the second term on the right-hand side of Eqs. (1) and (2) are wrong. Their signs should be just the opposite. In his notation,[2] there is a sign error in $\gamma(X)$ and consequently in ϕ'_X which may be verified easily by checking his calculations. An alternative way of obtaining the appropriate equations is discussed below.

Alternative Deduction

We start from the integral equation governing the reduced perturbation potential $\Phi(X,Y)$:

$$\Phi(X,Y)=\frac{1}{2\pi}\int_0^1\Big[\psi(X,\xi;Y,0)\Delta\Phi_\eta(\xi)$$

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*Reader, Department of Mathematics.

$$-\psi_\eta(X,\xi;Y,0)\Delta\Phi(\xi)\Big]d\xi+\frac{1}{2\pi}\int_S\int\psi(X,\xi;Y,\eta)$$

$$\times\Phi_\xi(\xi,\eta)\Phi_{\xi\xi}(\xi,\eta)dS+\text{const} \tag{8}$$

which is obtained[3] by inverting the transonic small-perturbation gasdynamic equation

$$\Phi_{XX}+\Phi_{YY}=\Phi_X\Phi_{XX} \tag{9}$$

by means of Green's theorem of potential theory, where

$$\psi(X,\xi;Y,\eta)=\ln[(\xi-X)^2+(Y-\eta)^2]^{1/2} \tag{10}$$

The corresponding integral equation governed by the Prandtl solution $\Phi_p(X,Y)$ is given by

$$\Phi_p(X,Y)=\frac{1}{2\pi}\int_0^1\Big[\psi(X,\xi;Y,0)\Delta\Phi_{p\eta}(\xi)$$

$$-\psi_\eta(X,\xi;Y,0)\Delta\Phi_p(\xi)\Big]d\xi+\text{const} \tag{11}$$

According to the thin-airfoil theory, boundary conditions are

$$\Delta\Phi_Y(X)=\Delta\Phi_{pY}(X),\quad 0\leq X\leq 1 \tag{12}$$

Then, subtracting Eq. (11) from Eq. (8), it follows that

$$\Phi(X,Y)=\Phi_p(X,Y)-\frac{1}{2\pi}\int_0^1\psi_\eta(X,\xi;Y,0)\Big\{\Delta\Phi(\xi)$$

$$-\Delta\Phi_p(\xi)\Big\}d\xi+I(X,Y;U) \tag{13}$$

where the double integral $I(X,Y;U)$ is given by Eq. (6). Integrating by parts the second term on the right hand side of Eq. (13) and subsequently differentiating both sides of Eq. (13) with respect to X yields

$$U(X,Y)=U_p(X,Y)+\frac{1}{2\pi}\Big[\Big\{\Delta\Phi(1)-\Delta\Phi_p(1)\Big\}\frac{-Y}{(1-X)^2+Y^2}$$

$$+\Big\{\Delta\Phi(0)-\Delta\Phi_p(0)\Big\}\frac{Y}{X^2+Y^2}\Big]+\frac{1}{2\pi}\int_0^1\Big\{\Delta U(\xi)$$

$$-\Delta U_p(\xi)\Big\}\frac{Y}{(\xi-X)^2+Y^2}d\xi+\frac{\partial I}{\partial X} \tag{14}$$

Taking the limit as $Y\to 0$, the term in the brackets vanishes, and we get

$$U(X,+0)=U_p2(X,+0)+\lim_{Y\to+0}\frac{1}{2\pi}\int_0^1\Big\{\Delta U(\xi)-\Delta U_p(\xi)\Big\}$$

$$\times\frac{Y\,d\xi}{(\xi-X)^2+Y^2}+\lim_{Y\to+0}\frac{\partial I}{\partial X} \tag{15}$$

To evaluate the second term on the right-hand side of Eq. (15), it is to be noted that the only nonzero contribution to the integral can come from the singularity at $\xi=X$, $Y=0$, and we see that

$$\lim_{Y\to+0}\frac{1}{2\pi}\int_0^1\Big\{\Delta U(\xi)-\Delta U_p(\xi)\Big\}\frac{Y\,d\xi}{(\xi-X)^2+Y^2}$$

$$=\frac{\Delta U(X)-\Delta U_p(X)}{2}\Big[\lim_{Y\to+0}\int_0^1\frac{Y\,d\xi}{(\xi-X)^2+Y^2}\Big]$$

$$=\tfrac{1}{2}[\Delta U(X)-\Delta U_p(X)] \tag{16}$$

Substituting Eq. (16) in Eq. (15) it follows that

$$U(X,+0)=U_p(X,+0)+\frac{1}{2}\Big[\Delta U(X)-\Delta U_p(X)\Big]+\lim_{Y\to+0}\frac{\partial I}{\partial X} \tag{17}$$

Similarly, taking the limit $Y\to-0$ in Eq. (14), it follows that

$$U(X,-0)=U_p(X,-0)-\frac{1}{2}\Big[\Delta U(X)-\Delta U_p(X)\Big]+\lim_{Y\to-0}\frac{\partial I}{\partial X} \tag{18}$$

The second term on the right of Eqs. (17) and (18) differs only in the sign from the corresponding terms of Eqs. (1) and (2). It should be mentioned that the corresponding terms in the three-dimensional formulation[4] have correct signs. However, these equations are not independent. For, subtracting Eq. (18) from Eq. (17), it follows that

$$\Delta U(X)=\Delta U_p(X)+[\Delta U(X)-\Delta U_p(X)]$$

$$+\Big[\lim_{Y\to+0}\frac{\partial I}{\partial X}-\lim_{Y\to-0}\frac{\partial I}{\partial X}\Big] \tag{19}$$

Since Φ_X and Φ_{XX} are Hölder continuous in the simply connected plane domain S, it follows from well-known results of potential theory[10] that $\partial I/\partial X$ exists and is also Hölder continuous. It then is not difficult to see that

$$\lim_{Y\to+0}\frac{\partial I}{\partial X}=\lim_{Y\to-0}\frac{\partial I}{\partial X} \tag{20}$$

Consequently, Eq. (19) leads to an equation of the form "zero equal to zero," indicating that Eqs. (17) and (18) are not independent. Incidentally, the second independent equation is delivered by the velocity component V together with the tangency boundary condition at the profile.

Finally, it should be mentioned that Nixon[7] did not detect the sign error. Assuming Eqs. (1) and (2) to be correct, he tried to show that they lead to nonunique results. However, his proof is not convincing, since Eqs. (1) and (2) (the incorrect form) are actually independent. The reason for the nonunique result should be sought in the iteration procedure adopted by Nixon[7] for this purpose.

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Technical Comments

Comment on "Reply by Author to A. H. Flax"

Alexander H. Flax*
Institute for Defense Analyses, Arlington, Va.

CHOPRA,[1] in his Reply to my Comment[2] on his Note,[3] seems to have missed one of the main points of my remarks. He states that, for the special cases of supersonic panel flutter which he treated, his procedures given in Ref. 1 will lead to the same results as those indicated in Ref. 2, the two procedures being merely alternative ways of arriving at the same answer. This is not the case. In fact, Chopra's new interpretation[1] of the formulas he presented in Ref. 3 leads to a recipe which cannot be carried out at all.

The transformation given in Ref. 2 relating the curve of λ vs g_T for a panel unrestrained by an elastic foundation to the corresponding curve for a panel restrained by an elastic foundation of stiffness, $K=K_I$, is unique and one to one for corresponding points on the curves. The transformation depends on a knowledge not only of g_T, the damping coefficient, and λ, the flutter speed parameter, but also of ω_F, the flutter frequency. With this information, it is possible to relate points on the two curves at the same value of λ by the formula[2]

$$g_{Te}=g_T/(1+K_I/m\omega_F^2)^{1/2} \qquad (1)$$

where the subscript e refers to the panel on an elastic foundation. This transformation carries point A on the flutter boundary of the unrestrained panel to the point A' of the flutter boundary of the elastically restrained panel, as shown in Fig. 1.

If point A is, in fact, the flutter point of the unrestrained panel (corresponding to a given value of g_T) and if the only change in the physical parameters of the panel is the addition of the restraint of an elastic foundation, the flutter point of the restrained panel is the point B' on the transformed curve. However, there is no way to proceed directly from point A to point B'. Instead, the transformation to B' must be from point B. One cannot use Eq. (1) or any formula given by Chopra[3] to determine the value of damping corresponding to

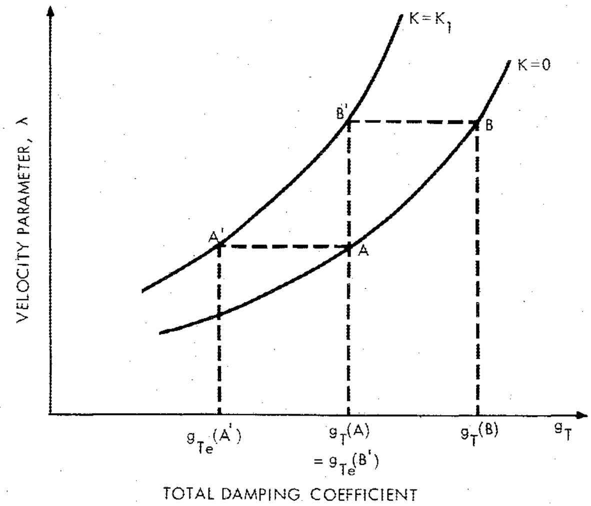

Fig. 1 Transformation of stability boundary.

point B, since the flutter frequency for point B is not known in advance.

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*President. Fellow AIAA.

Comment on "Natural Frequencies of a Cantilever with Slender Tip Mass"

Alexander H. Flax*
Institute for Defense Analyses, Arlington, Va.

IN Ref. 1, Bhat and Kulkarni have compared a perturbation solution attributed to Bhat and Wagner with the exact solutions for the vibration frequencies of a uniform cantilever beam with a tip mass M having an angular moment of inertia about its center of gravity J_0 for which the center of gravity may be displaced from the point of attachment of the beam by a distance L. Forming nondimensional quantities in terms of the length of the cantilever ℓ and its mass per unit length m,

$$\alpha=M/m\ell, \quad \epsilon=L/\ell, \quad \beta=J_0/m\ell^3 \qquad (1)$$

are defined. The unperturbed problem has given α and β with $\epsilon=0$; ϵ is the perturbation parameter.

The authors of Ref. 1 conclude that the perturbation method, including terms up to the second order, gives good results except for the case $\beta=0$. Unfortunately, their formulas for the perturbation method appear to be in error, and this, in turn, gives rise to large errors in their numerical results for the perturbation method in the case $\beta=0$, leading finally to their incorrect conclusion that the perturbation method fails in some way for $\beta=0$. The first-order errors in the numerical

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